ISRAEL JOURNAL OF MATHEMATICS, Vol. 65, No. 1, 1989

INDEPENDENCE RESULTS FOR UNCOUNTABLE SUPERSTABLE THEORIES

BY
LUDOMIR NEWELSKI
Mathematical Institute of the Polish Academy of Sciences, Kopernika 18, 51-617 Wroclaw, Poland

ABSTRACT
We prove that the following statement is independent of ZFC + $\neg$CH: If T is a superstable theory of power $< 2^{\aleph_0}$, $M \not\prec N$ are models of T with $Q(M) = Q(N)$, then there is $N' \not\succ N$ with $Q(N) = Q(N')$. This generalizes Lachlan's (1972) result.

§0. Introduction

In [L], A. H. Lachlan proved the following three theorems for countable stable T.

(LA) If $A \subseteq \mathfrak{C}$, then there is a model M of T such that $A \subseteq M$ and M is locally atomic over A.

(EP) If $(Q(x), A)$ has $T-V$ property, then there is a model M of T containing A such that $Q(M) = Q(A)$.

(ME) If $M \not\prec N$ are models of T with $Q(M) = Q(N)$, then there is $N' \not\succ N$ with $Q(N') = Q(N)$.

In this paper we investigate how far we can extend these theorems for uncountable stable theories. We use standard notation, such as can be found in [S]. Throughout, T (with possible subscripts) denotes a first-order theory in a language $L = L(T)$, M, N are models of T, $Q(x)$ is a predicate of $L(T)$. As usual, we assume that all models of T under consideration are elementary submodels of a fixed monster model $\mathfrak{C}$. If $p(x)$ is a type of T and $A \subseteq \mathfrak{C}$, then

Received May 14, 1987 and in revised form August 21, 1988

$p(A)=\{a\in A: a$ realizes $p\}$. Formulas are special cases of types. For $A\subseteq \mathfrak{C}$, $L(A)$ is the set of formulas of L with parameters from A. If $A=\varnothing$, then we omit it in $L(A)$. If $\varphi(x;\bar{y})\in L$ and p is a type of T, then

$$p\restriction\varphi=\{\chi(x)\in p:\chi(x)=\varphi(x;\bar{m}) \text{ or } \chi(x)=\neg\varphi(x;\bar{m}) \text{ for some } \bar{m}\in\mathfrak{C}\}.$$

For a formula $\theta(x)\in L(\mathfrak{C})$, $[\theta]$ is the class of types of T containing θ. A type p over A is locally isolated over $B\supseteq A$ if for every $\varphi(x;\bar{y})\in L$ there is $\chi(x)\in L(B)$ such that $\{\chi(x)\}\cup p(x)$ is consistent and $\chi(x)\vdash p\restriction\varphi$. A model M of T containing A is locally atomic over A if every $\bar{m}\in M$ realizes over A a locally isolated (complete) type. The notion of a locally atomic model is due to Shelah (F^l-atomic in [S]). For a finite set $\Delta\subseteq L$ we define $p\restriction\Delta$ as $\bigcup\{p\restriction\varphi:\varphi\in\Delta\}$. By [S, Lemma III, 2.1], the difference between a formula φ and a finite set of formulas Δ is negligible. D denotes Shelah's degree defined on formulas. If $\psi(x;\bar{y})\in L$, then for $\varphi(x)\in L(\mathfrak{C})$, $R-M(\varphi(x);\psi(x;\bar{y}))$ is the pair: $(\{\psi(x;\bar{y}),x=y\}$ — Morely rank of $\varphi(x)$, $\{\psi(x;\bar{y}),x=y\}$ — multiplicity of $\varphi(x))$. We also define $R_2(\varphi(x);\psi(x;\bar{y}))$ as Shelah's binary ψ-rank of $\varphi(x)$. We order $\omega\times\omega$ by $<$ lexicographically. In general we shall not use any tools of stable model theory developed after 1972. So in particular the acquaintance with forking is not necessary. This makes this paper accessible for wider range of readers.

Let us present the set-theoretical background. We work in ZFC. $\mathfrak{M}$, $\mathfrak{N}$ denote countable transitive models of ZFC, and we consider T, M, L and so on as elements of these models.

Let cov **K** be the minimal number of meager sets necessary to cover the real line. Let cov L be the minimal number of sets of Lebesgue measure zero necessary to cover the real line. κ_1 is the minimal power of a partition of the real line into compact sets. For $f,g\in{}^\omega\omega$ we define $f\text{-}3g$ iff for all but finitely many n, $f(n)<g(n)$.

We define

$$\mathfrak{b}=\min\{|F|:F\subseteq{}^\omega\omega\ \&\ \forall g\in{}^\omega\omega\ \exists f\in F\ \neg\, f\text{-}3g\}$$

and

$$\mathfrak{d}=\min\{|F|:F\subseteq{}^\omega\omega\ \&\ \forall g\in{}^\omega\omega\ \exists f\in F\ g\text{-}3f\}.$$

We have $\aleph_1\leqq\mathfrak{b}$, cov **K**, cov **L**; $\mathfrak{b}+\text{cov}\,\mathbf{K}\leqq\mathfrak{d}\leqq\kappa_1\leqq 2^{\aleph_0}$. The reader may find more information on these coefficients for example in [K], [M1], [M2] or [N]. MA, CH denote Martin's axiom and continuum hypothesis, respectively.

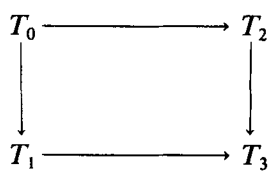
For $A\subseteq\mathfrak{C}$ we say that $(Q(x),A)$ has T–V property if for every $\theta(x)\in L(A)$ with $\theta(x)\vdash Q(x)$, if $\theta(x)$ is consistent then there is $a\in A$ such that $\theta(a)$ holds.

We say that T has *extension property* if T satisfies (EP). T has *model extension property* if T satisfies (ME).

Let us go back to Lachlan's theorems. Since Lachlan proved them, some people tried to generalize them for uncountable stable theories. The attention was focused particularly on (ME). V. Harnik in [H] really proved it for uncountable stable theories (slightly generalizing Lachlan's proof), however he added additional assumption that M is $|T|$-compact. J. Baldwin in [B] gave another proof of (ME) for countable stable T (see also [Ls]). In this paper we show that there are stable theories of power $\aleph_1$ without model extension property, so neither of (LA), (EP), (ME) can be proved for uncountable stable theories. However, the situation for superstable theories is quite different. We prove for example that it is consistent with ZFC $+ \neg$CH that every superstable T of power $< 2^{\aleph_0}$ satisfies (LA), (EP) and (ME). Also we show that it may be the case that $\neg$CH holds and there is a superstable T of power $\aleph_1$ without model extension property. In the first section we present examples, in the second one theorems.

§1. Examples

We define here 4 uncountable stable theories T_0, T_1, T_2, T_3. The following diagram visualizes connections between them.



An arrow from T_i to T_j indicates that T_j is similar to T_i, but more complicated, and if two arrows are parallel then the complication has the same character.

EXAMPLE 0. We construct here a superstable theory T_0 of power $2^{\aleph_0}$ without extension property. The set A in (EP) is countable here and $Q(x)$ is even strongly minimal. T_0 is a variant of the theory from ex. IV, 2.13(3) in [S]. The language of T_0 consists of unary predicates $Q(x)$, $P_\eta(x)$ for $\eta \in {}^{\omega>}2$, unary function symbols f_η for $\eta \in {}^{\omega}2$ and constants m_n for $n < \omega$. The axioms of T_0 are

0.1. All the predicates are consistent.

0.2. $P_\varnothing(x) \dot\vee Q(x)$.

0.3. $P_\eta(x) \leftrightarrow P_{\eta^\frown\langle 0\rangle}(x) \dot\vee P_{\eta^\frown\langle 1\rangle}(x)$.

0.4. f_η is a function mapping $P_\varnothing$ onto Q and for every y satisfying Q, $f_\eta^{-1}(\{y\})$ is infinite.

0.5. $Q(m_n)$ for $n < \omega$.

0.6. If $\nu_0 \lessdot \eta \in {}^\omega 2$, $|\nu_0| = n$ and $\nu = \nu_0{}^\frown\langle 1 - \eta(n)\rangle$, then

$$P_\nu(x) \leftrightarrow f_\eta(x) = m_n$$

is an axion of T_0.

Figure 1 shows the action of f_η on $P_\varnothing$ for $\eta \equiv 0$. For other η's the corresponding picture would look similar.

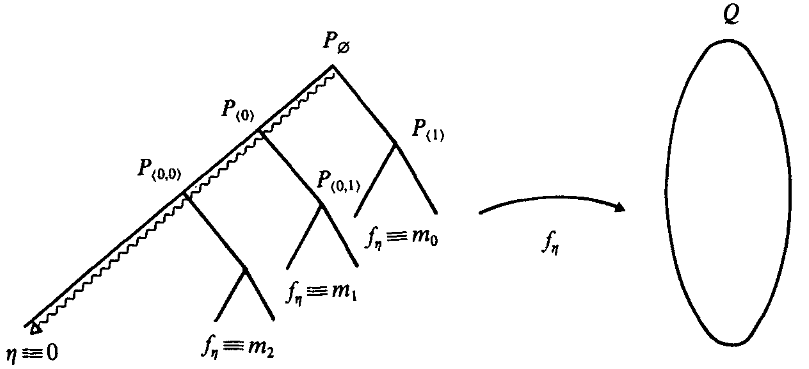


Fig. 1.

Clearly T_0 is a complete, superstable theory. Moreover, $Q(x)$ is strongly minimal. Let $A = \{m_n : n < \omega\}$. Of course $(Q(x), A)$ has T–V property. Assume that $M \models T_0$. Let $a \in P_\varnothing(M)$. Choose the unique $\eta \in {}^\omega 2$ such that for $n < \omega$, $P_{\eta\restriction n}(a)$ holds in M. So 0.6 implies that $f_\eta(a) \neq m_n$ for every n. This means that $Q(M) \neq Q(A)$.

EXAMPLE 1. A superstable T_1 of power $2^{\aleph_0}$ without model extension property. The language of T_1 consists of

(a) unary predicates $Q(x)$, $P(x)$, $P_\eta^i(x)$ for $\eta \in {}^{\omega>}2$, $i < \omega$,

(b) constants m_n for $n < \omega$,

(c) unary function symbols F and f_η^i for $\eta \in {}^\omega 2$, $0 < i < \omega$,

(d) ternary predicate symbols $P^i(x, y, z)$ for $i < \omega$.

The main idea underlying the definition of T_1 consists in imitating within a model of T_1 many "partial" versions of T_0. The pair of models M, N in (ME) will satisfy $Q(M) = Q(N) =$ the set of constants of $L(T_1)$. The superscripts

$i < \omega$ are added to make the structure of T_1 on P outside the $P^i_\varnothing$'s trivial. Instead of $P^i(x, y, z)$ we shall write sometimes $P^i_{xy}(z)$, regarding it as a formula with variable z and parameters x, y. The axioms of T_1 are:

1.1. All the predicates of T_1 are consistent.

1.2. $P(x) \dot{\vee} Q(x)$.

1.3. $P^i_\varnothing(x) \rightarrow P(x)$; $P^i_\varnothing(x) \rightarrow \neg P^j_\varnothing(x)$ for $i \neq j$.

1.4. $P^i_\eta(x) \leftrightarrow P^i_{\eta^\frown\langle 0\rangle}(x) \dot{\vee} P^i_{\eta^\frown\langle 1\rangle}(x)$.

1.5. $Q(m_n)$ for $n < \omega$.

1.6. F is a function mapping P onto Q; pre-image of any point in Q by F is infinite.

1.7. $F(x) = m_n \leftrightarrow P^n_\varnothing(x)$.

1.8. $P^i(x, y, z) \rightarrow P^i_\varnothing(x) \,\&\, P^i_\varnothing(y) \,\&\, P^{i+1}_\varnothing(z) \,\&\, x \neq y \,\&\, P^i(y, x, z)$.

1.9. $(\forall x, y)(P^0_\varnothing(x) \,\&\, P^0_\varnothing(y) \,\&\, x \neq y \rightarrow \exists z\, P^0(x, y, z))$.

1.10. For $i > 0$, $(\forall x, y)(x \neq y \,\&\, (\exists v, t)(P^{i-1}(v, t, x) \,\&\, P^{i-1}(v, t, y)) \leftrightarrow (\exists z)(P^i(x, y, z)))$.

1.11. For $i > 0$, $(\forall x)(P^i_\varnothing(x) \rightarrow (\exists v, t)(P^{i-1}(v, t, x))$.

1.12. For $\eta \in {}^{\omega >}2$, $i < \omega$, $(\forall x, y)(\exists z\, P^i(x, y, z) \rightarrow \exists z\, P^i(x, y, z) \,\&\, P^{i+1}_\eta(z))$.

Up to now we have built a general frame of T_1. It is presented in Figure 2.

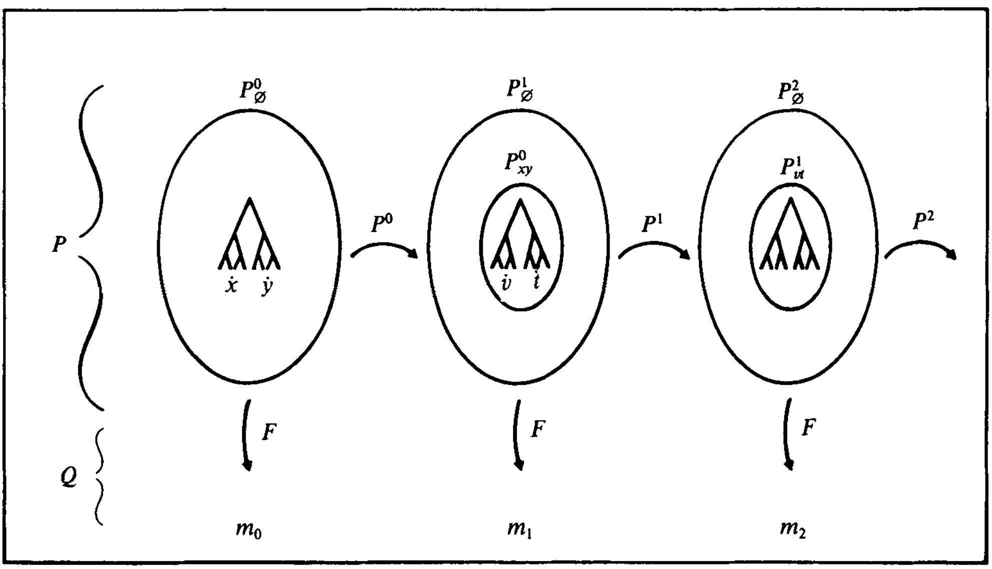


Fig. 2.

We are left with determining properties of the f^i_η's. First of all

1.13. f^i_η maps $P^i_\varnothing$ onto Q; the pre-image of any point in Q by f^i_η is infinite.

On every P^i_{xy} functions f^{i+1}_η will be defined to imitate T_0. If we imitated it too faithfully, then there would be no model M of T_1 satisfying $Q(M)=\{m_n : n<\omega\}$. This will be the point in proving $\neg$(ME). The degree of similarity between T_0 and T_1 on P^i_{xy} will depend on the degree of similarity between x and y in the tree $\{P^i_\eta : \eta \in {}^{\omega>}2\}$. Let us introduce the following abbreviation for x, y satisfying $P^i_\varnothing$:

$$(x\restriction n = y\restriction n) \leftrightarrow \bigwedge\{P^i_\eta(x)\leftrightarrow P^i_\eta(y) : \eta\in {}^n2\},$$

$$(x\restriction n = y\restriction n \mathrel{\&} x(n)\neq y(n)) \leftrightarrow (x\restriction n = y\restriction n) \mathrel{\&} \neg(x\restriction(n+1) = y\restriction(n+1)).$$

Here are the axioms describing the f^i_η's.

1.14. If P^{i-1}_{xy} is consistent, then f^i maps P^{i-1}_{xy} onto Q; the pre-image of any point in Q by f^i is infinite on P^{i-1}_{xy}.

1.15. Let $i, n<\omega$, $\eta\in{}^\omega 2$, $\nu_0 \lessdot \eta$, $|\nu_0| = n$, $\nu = \nu_0{}^\frown\langle 1-\eta(n)\rangle$. Then the following is an axiom of T_1:

$$(\forall x, y)(\exists z\, P^i(x,y,z) \mathrel{\&} x\restriction n = y\restriction n \to (\forall z\, P^i_{xy}(z)\to f^{i+1}_\eta(z) = m_n \leftrightarrow P^{i+1}_\nu(z))).$$

1.16. Let $i, k, t<\omega$, $\eta, \nu\in{}^\omega 2$.

(a) If $T_0 \vdash$ "$f^{-1}_\eta(\{m_k\}) \subseteq f^{-1}_\nu(\{m_t\})$", then the following is an axion of T_1:

$$(\forall x, y)(\exists z\, P^i(x,y,z) \to (\forall z)(P^i_{xy}(z) \mathrel{\&} f^{i+1}_\eta(z) = m_k \to f^{i+1}_\nu(z) = m_t)).$$

(b) If $T_0 \vdash$ "$(\forall y\neq m_0,\dots,m_k)(Q(y)\to f^{-1}_\eta(\{y\})\subseteq f^{-1}_\nu(\{m_t\}))$", then the following is an axiom of T_1:

$$(\forall x, y)(\exists z\, P^i(x,y,z) \to (\forall z)(P^i_{xy}(z) \mathrel{\&} f^{i+1}_\eta(z)\neq m_0,\dots,m_k \to f^{i+1}_\nu(z) = m_t)).$$

1.17. Let $i, n, k<\omega$, $\eta\in{}^\omega 2$, $\nu\in{}^\omega 2$, $|\nu|>n$ and

$$\mathrm{T}_0 \vdash (\forall y\neq m_0,\dots,m_k)(Q(y)\to(\exists x)(P_{\nu\restriction n+1}(x) \mathrel{\&} f_\eta(x) = y)).$$

Then the following is an axiom of T_1:

$$(\forall x, y)(\forall t\neq m_0,\dots,m_k)(\exists z\, P^i(x,y,z) \mathrel{\&} Q(t) \mathrel{\&} x\restriction n = y\restriction n \mathrel{\&} x(n)\neq y(n)$$

$$\to \exists z\, P^i_{xy}(z) \mathrel{\&} P^{i+1}_\nu(z) \mathrel{\&} f^{i+1}_\eta(z) = t).$$

1.1–1.17 are all axioms of T_1. Roughly speaking, 1.15 determines similarity between T_0 and T_1 on P^i_{xy} "up to level n" if $x\restriction n = y\restriction n$, while 1.16 and 1.17

complete the structure on P^i_{xy} in such a way that "nothing new" can be said about Q.

Once more T_1 is superstable and $Q(x)$ is strongly minimal. Now we prove $\neg$(ME). There is a countable model N of T_1 such that $Q(N)=\{m_n : n<\omega\}$. In order to construct such a model it is sufficient to ensure that for every $x\neq y\in N$ there is $n<\omega$ such that $x\restriction n=y\restriction n$ does not hold in N. Thus there is also $N'\not\cong N$ with $Q(N')=Q(N)$. To complete the proof it suffices to prove that there is no model M of T_1 of power $>2^{\aleph_0}$ with $Q(M)=Q(N)$. Suppose to the contrary that M is such a model. First, because of 1.7, $P(M)=\bigcup\{P^i_\varnothing(M) : i<\omega\}$. Choose the minimal $i<\omega$ such that $|P^i_\varnothing(M)|>2^{\aleph_0}$. If $i=0$, then by 1.12 we get v,t such that $|P^{i-1}_{vt}(M)|>2^{\aleph_0}$, and therefore we can pick $x\neq y\in P^{i-1}_{vt}(M)$ such that for every $n<\omega$, $x\restriction n=y\restriction n$. Anyway, we get x,y such that $P^i_{xy}(M)\neq\varnothing$ and $x\restriction n=y\restriction n$ for each n. So 1.14 determines on $P^i_{xy}(M)$ a structure similar to that of T_0. Let $a\in P^i_{xy}(M)$. Choose $\eta\in{}^\omega 2$ such that for every $n<\omega$, $P^{i+1}_{\eta\restriction n}(a)$ holds. 1.14 implies that $f^{i+1}_\eta(a)\neq m_n$ for every n, thus $Q(M)\neq\{m_n : n<\omega\}$. This means that $\neg$(ME) is established.

T_0 and T_1 are superstable of power $2^{\aleph_0}$. We shall see in the next section that in ZFC it is impossible to find superstable T without extension property and of power $<2^{\aleph_0}$. However, if we add some extra axioms to ZFC, we can find such a T. Below we show how it can be done.

In the construction of T_0 we connected with every f_η the forbidden set $\{\eta\}\subseteq{}^\omega 2$ with the property that for any a realizing $\{P_{\eta\restriction n}(x) : n<\omega\}$ we had $f_\eta(a)\neq m_n$, $n<\omega$. Clearly $\{\eta\}$ can be replaced by any nowhere dense closed set $N\subseteq{}^\omega 2$. Let $\{N_\alpha : \alpha<\kappa_1\}$ be a family of NWD closed disjoint sets covering ${}^\omega 2$. We can construct T_0' in such a way that instead of functions $\{f_\eta : \eta\in{}^\omega 2\}$ we have functions $\{f_\alpha : \alpha<\kappa_1\}$, and f_α is connected with N_α in the same way as f_η is connected with $\{\eta\}$. $Q(x)$ will no longer be strongly minimal, but T_0' will still be superstable (one can split $Q(x)$ into predicates $Q_\alpha(x)$, $\alpha<\kappa_1$). If we go from T_0' to T_1' in such a way as we went from T_0 to T_1, we get a superstable T_1' of power κ_1 without model extension property. It is well known that ZFC $+\ \kappa_1=\aleph_1+$ "$2^{\aleph_0}$ large" is relatively consistent (see [M1] or [N]).

EXAMPLE 2. We construct here a stable T_2 of power $\aleph_1$ without extension property. The set A in (EP) has power $\aleph_1$ here. This is a preparatory step in constructing a stable T_3 of power $\aleph_1$ without model extension property. T_2 will be similar to T_0, but more complicated. We can use only ω_1-many functions f_α, so we have to "embed" into T_2 a topological space which is a union of ω_1-many NWD closed sets. A good candidate for such a space is ${}^\omega\omega_1$ with product topology. Let $N_\alpha = \{f \in {}^\omega\omega_1 : \forall n\, f(n) < \alpha\}$. Clearly N_α is closed and NWD for $\alpha < \omega_1$ and ${}^\omega\omega_1$ is the increasing union of N_α, $\alpha < \omega_1$. On the other hand, we want to have a possibility to imitate fragments of T_2 in T_3 with a certain prescribed degree of faithfulness. This is why we use functions instead of predicates to describe ${}^\omega\omega_1$. The language of T_2 consists of

(a) unary predicates $P(x)$, $Q(x)$, $Q^n(x)$ for $n > 0$ and $Q^{\omega\alpha}(x)$ for $0 < \alpha < \omega_1$,

(b) constants m^n_η for $0 < n < \omega$, $\eta \in {}^n\omega_1$,

(c) constants $m^{\omega\alpha}_\eta$ for $0 < \alpha < \omega_1$ and $\eta \in {}^{\omega>}\omega_1$ satisfying $\eta \restriction |\eta| - 1 \in {}^{\omega>}\omega \cdot \alpha$ and $\eta(|\eta| - 1) \geqq \omega \cdot \alpha$,

(d) unary function symbols f^n for $0 < n < \omega$ and $f^{\omega\alpha}$ for $0 < \alpha < \omega_1$.

Instead of writing down all axioms of T_2, which might be rather tedious, we determine T_2 by exhibiting models of T_2 restricted to sublanguages of $L(T_2)$ with only countably many function symbols. For $\alpha < \omega_1$ let L_α consist of all predicates and constants of $L(T_2)$, unary function symbols f^n for $0 < n < \omega$ and $f^{\omega\beta}$ for $0 < \beta \leqq \alpha$. We show a model M_α of $T_2 \restriction L_\alpha$. The universe of M_α is

$$(\text{the set of constants of } L(T_2)) \,\dot\cup\, \{f \in {}^\omega\omega_1 : \exists n\, f(n) \geqq \omega\alpha\}.$$

Let $Q(M_\alpha)$ be the set of constants of $L(T_2)$, $P(M_\alpha) = |M_\alpha| - Q(M_\alpha)$ and we define all other symbols of L_α on M_α so that the following hold.

2.1. $Q^n(M_\alpha)$ for $0 < n < \omega$ and $Q^{\omega\beta}(M_\alpha)$ for $0 < \beta \leqq \alpha$ are all pairwise disjoint and contained in $Q(M_\alpha)$.

2.2. $Q^n(m^n_\eta)$, $Q^{\omega\beta}(m^{\omega\beta}_\eta)$.

2.3. f^n, $f^{\omega\beta}$ are functions mapping $P(M_\alpha)$ onto $Q^n(M_\alpha)$, $Q^{\omega\beta}(M_\alpha)$ respectively.

2.4. For $\eta \in {}^n\omega_1$, $\nu \in {}^k\omega_1$ with $\nu \lessdot \eta$, $(\forall x)(f^n(x) = m^n_\eta \rightarrow f^k(x) = m^k_\nu)$ holds in M_α.

2.5. For $\eta \in {}^n\omega_1$ such that $m^{\omega\beta}_\eta$ is a constant of $L(T_2)$, $f^{\omega\beta}(x) = m^{\omega\beta}_\eta \leftrightarrow f^n(x) = m^n_\eta$ holds in M_α.

Let $g \in P(M_\alpha)$. For $0 < n < \omega$ we define simply $f^n(g) = m^n_{g \restriction n}$. Let $0 < \beta \leqq \alpha$. Choose the minimal $k < \omega$ such that $g(k) \geqq \omega\beta$. We define $f^{\omega\beta}(g) = m^{\omega\beta}_{g \restriction k+1}$. It is tedious but standard to check that $\mathrm{Th}(M_\alpha)$ is stable and $\mathrm{Th}(M_\alpha) \subseteq \mathrm{Th}(M_\beta)$ for $\alpha < \beta < \omega_1$. Thus we can define

$$T_2 = \bigcup \{\mathrm{Th}(M_\alpha) : \alpha < \omega_1\}.$$

Let A be the set of constants of $L(T_2)$. It is easy to see that $(Q(x), A)$ has T–V property. We need only check that there is no model M of T_2 with $Q(M) = Q(A)$. Suppose that M is such a model. Let $a \in P(M)$. For every $n > 0$ we have

$$P(M) = \bigcup \{(f^n)^{-1}(\{m^n_\eta\}) : \eta \in {}^n\omega_1\}.$$

Thus there is $\eta \in {}^\omega\omega_1$ such that for every $n > 0$, $f^n(a) = m^n_{\eta \restriction n}$. Let β be any ordinal $< \omega_1$ such that for every $n < \omega$, $\eta(n) < \omega\beta$. So by 2.5 we have $f^{\omega\beta}(a) \notin Q(A)$, because for no $n > 0$, $m^{\omega\beta}_{\eta \restriction n}$ is a constant of $L(T_2)$.

Example 3 of a stable T_3 of power $\aleph_1$ without model extension property. The transition from T_2 to T_3 is similar to that from T_0 to T_1. We shall state explicitly only some axioms of T_3, and then construct a model of T_3. $L(T_3)$ consists of

(a) unary predicates $Q(x)$, $Q^n(x)$ for $n < \omega$, $Q^{\omega\alpha}(x)$ for $0 < \alpha < \omega_1$, $V(x)$ and $V^i(x)$ for $i < \omega$,

(b) constants of $L(T_2)$ and m^0_n for $n < \omega$,

(c) unary function symbols F, f^n_i for $0 < n < \omega$, $i < \omega$ and $f^{\omega\alpha}_i$ for $0 < \alpha < \omega_1$, $0 < i < \omega$,

(d) ternary predicate symbols $P^i(x, y, z)$ for $i < \omega$.

The pair of models M, N in (ME) will satisfy $Q(M) = Q(N) =$ the set of constants of $L(T_3)$. As in Example 1, instead of $P^i(x, y, z)$ we shall write sometimes $P^i_{xy}(z)$. Here are some axioms of T_3.

3.1. All the predicates of T_3 are consistent.

3.2. $V(x) \dot{\vee} Q(x)$.

3.3. V^i for $i < \omega$ are pairwise disjoint and imply V.

3.4. Q^n for $n < \omega$, $Q^{\omega\alpha}$ for $0 < \alpha < \omega_1$ are pairwise disjoint and imply Q.

3.5. $Q^0(m^0_i)$, $Q^n(m^n_\eta)$ for $n > 0$, $Q^{\omega\alpha}(m^{\omega\alpha}_\eta)$.

3.6. F is a function mapping V onto Q^0.

3.7. $F(x) = m^0_n \leftrightarrow V^n(x)$.

3.8. $P^i(x, y, z) \to V^i(x)\ \&\ V^i(y)\ \&\ V^{i+1}(z)\ \&\ x \neq y\ \&\ P^i(y, x, z)$.

3.9. $(\forall x, y)(V^0(x)\ \&\ V^0(y)\ \&\ x \neq y \to \exists z\, P^0(x, y, z))$.

3.10. For $i > 0$, $(\forall x, y)(x \neq y\ \&\ (\exists v, t)(P^{i-1}(v, t, x)\ \&\ P^{i-1}(v, t, y)) \leftrightarrow \exists z\, P^i(x, y, = z))$.

3.11. For $i > 0$, $(\forall x)(V^i(x) \to (\exists v, t)(P^{i-1}(v, t, x)))$.

Let L' be the sublanguage of $L(T_3)$ consisting of all its predicates, constants and function symbol F. Let M' be a model of 3.1–3.11 such that $Q(M')$ is the set of constants of $L(T_3)$ and

(1) for every $x, y \in M'$, if $P^i_{xy}(M') \neq \varnothing$ then $|P^i_{xy}(M')| = 2^{\aleph_0}$,

(2) $|V^0(M')| = 2^{\aleph_0}$.

We shall expand M' to a model for $L(T_3)$. f^n_i for $n > 0$, $i < \omega$ and $f^{\omega\alpha}_i$ for $0 < \alpha < \omega_1$, $i > 0$, will be defined so that

3.12. For every $x, y \in M'$, if $P^i_{xy}(M') \neq \emptyset$ then $f^n_{i+1}, f^{\omega\alpha}_{i+1}$ map $P^i_{xy}(M')$ onto Q^n, $Q^{\omega\alpha}$ respectively.

3.13. f^n_0 maps V^0 onto Q^n.

First we define f^n_i on M' so that

(3) for every $x, y \in M'$, if $P^i_{xy}(M') \neq \emptyset$ then for every $\eta \in {}^\omega\omega_1$ there is exactly one $z \in P^i_{xy}(M')$, and for every $z \in P^i_{xy}(M')$ there is $\eta \in {}^\omega\omega_1$, such that for $n > 0$, $M' \models f^n_{i+1}(z) = m^n_{\eta \upharpoonright n}$,

(4) for every $\eta \in {}^\omega\omega_1$ there is exactly one $z \in V^0(M')$ and for every $z \in V^0(M')$ there is $\eta \in {}^\omega\omega_1$ such that for $n > 0$, $M' \models f^n_0(z) = m^n_{\eta \upharpoonright n}$.

Fix $x, y \in M'$ such that $P^i_{xy}(M') \neq \emptyset$. The only thing left is to define functions $f^{\omega\alpha}_{i+1}$ for $0 < \alpha < \omega_1$ on $P^i_{xy}(M')$. As x, y are fixed, we can drop the index i in f^n_{i+1}, $f^{\omega\alpha}_{i+1}$ and P^i_{xy}. We have already embedded ${}^\omega\omega_1$ into P_{xy}. Let $k < \omega$ be minimal such that $f^k_i(x) \neq f^k_i(y)$ holds in M'. $f^{\omega\alpha}$'s are chosen so that the following hold.

3.14. For $n \leqq k$ and $\eta \in {}^n\omega_1$ such that $m^{\omega\alpha}_\eta$ is a constant of $L(T_3)$, $(f^{\omega\alpha}(z) = m^{\omega\alpha}_\eta \leftrightarrow f^n(z) = m^n_\eta)$ holds in $P_{xy}(M') \cup Q(M')$.

3.15. For $\eta \in {}^k\omega_1$ and $\nu \in {}^{\omega>}\omega_1$, if $T_2 \vdash f^{\omega\alpha}(z) = m^{\omega\alpha}_\nu \rightarrow f^k(z) = m^k_\eta$ then for every $\eta' \in {}^l\omega_1$ with $\eta \lessdot \eta'$ we have in $P_{xy}(M')$:

$$(f^{\omega\alpha})^{-1}(\{m^{\omega\alpha}_\nu\}) \cap (f^l)^{-1}(\{m^l_{\eta'}\}) \neq \emptyset$$

and

$$(f^{\omega\alpha})^{-1}(\{m^{\omega\alpha}_\nu\}) \subseteq (f^k)^{-1}(\{m^k_\eta\}).$$

3.16. In $P_{xy}(M')$ we have $(f^{\omega\alpha})^{-1}(\{m^{\omega\alpha}_\eta\}) \subseteq (f^{\omega\beta})^{-1}(\{m^{\omega\beta}_\nu\})$ iff the same holds in any model of T_2.

Axiom 3.14 ensures that when $k \rightarrow \infty$ then the structure on $P_{xy}(M') \cup Q(M')$ converges to that of a model of T_2. 3.15 and 3.16 ensure that on $Q(M')$ no new connections arise. It is easy to see that we can define $f^{\omega\alpha}$ for $0 < \alpha < \omega_1$ on $P_{xy}(M')$ according to 3.12–3.16. Thus M' with the just defined functions becomes a structure M for $L(T_3)$. Let $T_3 = \mathrm{Th}(M)$. It is possible to realize that some definitional extension of T_3 admits elimination of quantifiers and that T_3 is stable. Now we shall show that T_3 does not have model extension property. First, exactly as in Example 1, we can find models $N \not\prec N'$ of T_3 such that $Q(N') = Q(N) =$ the set of constants of $L(T_3)$. Thus to prove $\neg$(ME) it suffices to observe that there is no model M of T_3 of power $> 2^{\aleph_0}$ with $Q(M) = Q(N)$. Suppose to the contrary that there is such an M. Then, because of 3.6 and 3.7, $V(M) = \bigcup \{V^i(M) : i < \omega\}$. Choose the minimal $i < \omega$ such that $|V^i(M)| > 2^{\aleph_0}$. As in Example 1 we conclude that there are $x, y \in V(M)$ such that $P^i_{xy}(M) \neq \emptyset$ and for every $n > 0$, $f^n_i(x) = f^n_i(y)$ holds. But now, by 3.14 we can proceed exactly as in Example 2 to get a contradiction.

§2. Theorems

This section stands in opposition to the previous one. Instead of constructing counterexamples to (LA), (EP) and (ME), we prove that (LA), (EP) and (ME) can be true for uncountable superstable theories of power $< 2^{\aleph_0}$. First let us notice the following

FACT 2.1. Let T be a stable theory.
(1) If T satisfies (LA) then T has extension property.
(2) If T has extension property then T has model extension property.

PROOF. (1) Suppose that $(Q(x), A)$ has T–V property. Let $M \supseteq A$ be a locally atomic over A model of T. Thus M omits $q(x) = \{Q(x),\ x \neq m\colon m \in Q(A)\}$. This means that $Q(M) = Q(A)$.

(2) Suppose that $M \not\prec N$ are models of T with $Q(M) = Q(N)$. Take $a \in N - M$ and $b \in \mathfrak{C}$ realizing over N the non-forking extension of $\mathrm{tp}(a/M)$ (those not familiar with forking can look at the relevant place in [L] on how to choose b). In [L] or [B] it is proved that $(Q(x), N \cup \{b\})$ has T–V property, so we are done.

The main result in this section is

THEOREM 2.2. *Assume that T is superstable, $A \subseteq \mathfrak{C}$ and one of* (A), (B), (C) *holds, where*
(A) $|T| < \operatorname{cov} \mathbf{K}$,
(B) $|T| < \mathfrak{b}$,
(C) $|T| < \min\{\operatorname{cov} \mathbf{L}, \mathfrak{d}\}$.
Then there is a model $M \supseteq A$ of T which is locally atomic over A.

REMARK. The model-theorist not interested in parts (B), (C) of Theorem 2.2 may omit reading the proofs of these parts. Part (A) is sufficient to draw Corollary 2.11 below. Parts (B), (C) are motivated by the attempt to find in Theorem 2.2 the largest possible cardinal with which to replace $\operatorname{cov} \mathbf{K}$. At present this cardinal is $\operatorname{cov} \mathbf{K} + \mathfrak{b} + \min\{\operatorname{cov} \mathbf{L}, \mathfrak{d}\}$ which is still $\leqq \mathfrak{d} \leqq \kappa_1$.

Clearly to prove Theorem 2.2 it suffices to prove

THEOREM 2.3. *Assume that T is superstable, $A \subseteq \mathfrak{C}$, $\theta(x)$ is a consistent formula from $L(A)$ and* (A), (B) *or* (C) *from Theorem 2.2 holds. Then there is a locally isolated $p \in [\theta] \cap S(A)$.*

PROOF OF 2.3. First, if there is an isolated $p \in [\theta] \cap S(A)$, we are done. So we may assume

(α) There is no isolated $p \in [\theta] \cap S(A)$.

For $\varphi(x; \bar{y}) \in L$ we define

$N(\varphi) = \{ p \in [\theta] \cap S(A)$: there is no $\chi(x) \in p(x)$ such that $\chi(x) \vdash p \restriction \varphi \}$.

Thus in order to prove Theorem 2.3 it suffices to show

(β) $[\theta] \cap S(A) \neq \bigcup \{N(\varphi) : \varphi \in L\}$.

FACT 2.4. For every $\varphi(x; \bar{y}) \in L$ there is $\Phi(x; \bar{z}_0) \in L$ such that for every $p \in N(\varphi)$ and for every $\Psi(x) \in p(x)$ there is $\bar{m} \in A$ such that $\Phi(x; \bar{m})$ & $\Psi(x)$ is consistent and $[\Phi(x; \bar{m})$ & $\Psi(x)] \cap N(\varphi) = \varnothing$. In particular $p(x) \vdash \neg \Phi(x; \bar{m})$.

PROOF. Let $n = R_2(x = x; \varphi(x; \bar{y})) + 1$. We define

$$\Phi(x; \bar{z}_0) = \bigwedge_{i<n} \varphi(x; \bar{y}_i^0) \;\&\; \bigwedge_{i<n} \neg \varphi(x; \bar{y}_i^1),$$

where we assume that $\bar{y}_i^0$, $i < n$, $\bar{y}_i^1$, $i < n$ are disjoint and $\bar{z}_0$ is their concatenation. Now let $\Psi(x) \in p(x)$. As $R_2(\Psi(x); \varphi(x; \bar{y})) < n$, we can find $\bar{m} \in A$ such that $\Phi(x; \bar{m})$ & $\Psi(x)$ is consistent and for every $\bar{a} \in A$, either $\varphi(x; \bar{a})$ & $\Phi(x; \bar{m})$ & $\Psi(x)$ or $\neg \varphi(x; \bar{a})$ & $\Phi(x; \bar{m})$ & $\Psi(x)$ is inconsistent. But this means that whenever $q(x) \in S(A) \cap [\Phi(x; \bar{m})$ & $\Psi(x)]$ then $\Phi(x; \bar{m})$ & $\Psi(x) \vdash q \restriction \varphi$, and so $q \notin N(\varphi)$.

REMARK. Fact 2.4 shows that $N(\varphi)$ is nowhere dense in $[\theta] \cap S(A)$. Thus if T were countable we could finish the proof with the remark that no compact space can be covered by countably many nowhere dense sets. However, there are compact spaces which can be covered by ω_1-many NWD sets, so this argument does not work in our case.

LEMMA 2.5. *Assume that $p \in N(\varphi)$. Then there is a formula $\chi(x; \bar{z}) \in L$ such that for every formula $\Psi(x) \in p(x)$ there are $\bar{m}_i \in A$ for $i < \omega$ such that for $i \neq j < \omega$, $\chi(x; \bar{m}_i)$ & $\Psi(x)$ is consistent and $\chi(x; \bar{m}_i) \vdash \neg \chi(x; \bar{m}_j)$.*

PROOF. Let $\Phi(x; \bar{z}_0)$ be the formula given by Fact 2.4. Let $L(\Phi)$ be the set of Boolean combinations of formulas $\Phi(x; \bar{z}_0^i)$, $i < \omega$, where we assume that $\bar{z}_0^i$, $i < \omega$, are pairwise disjoint. A typical element of $L(\Phi)$ can be written down as $\chi(x; \bar{\bar{z}}_0)$, where $\bar{\bar{z}}_0$ is a tuple of elements of $\{\bar{z}_0^i : i < \omega\}$.

Let (n, k) be $<$-minimal such that for some $\chi'(x; \bar{z}_0) \in L(\Phi)$, for every formula $\Psi(x) \in p(x)$ there is $\bar{m} \in A$ such that

(1) $R - M(\chi'(x; \bar{m}); \Phi(x; \bar{z}_0)) < (n, k)$,

(2) $\chi'(x; \bar{m})$ & $\Psi(x)$ is consistent, and

(3) $p(x) \vdash \neg\chi'(x; \bar{m})$.

By Fact 2.4 we see that there is $\chi'(x; \bar{z}_0)$ such that for every $\Psi(x) \in p(x)$ there is $\bar{m} \in A$ such that (1), (2), (3) hold for some (n_0, k_0). So (n, k) is the $<$-minimal element of some non-empty subset of $\omega \times \omega$.

Case 1. $k > 1$. So there is $\Psi_0(x) \in p(x)$ such that whenever $\bar{m} \in A$ satisfies (1), (2) and (3) with Ψ replaced by Ψ_0, then $R - M(\chi'(x; \bar{m}); \Phi) = (n, k-1)$. Consider

$$\chi_0(x; \bar{z}') = \chi'(x; \bar{z}_0) \ \&\ \chi'(x; \bar{z}_1) \quad \text{and} \quad \chi_1(z; \bar{z}') = \chi'(x; \bar{z}_0) \ \&\ \neg\chi'(x; \bar{z}_1),$$

where we assume that $\bar{z}_0$, $\bar{z}_1$ are disjoint and $\bar{z}'$ is their concatenation. By the minimality of (n, k), there is $\Psi_1(x) \in p(x)$ such that $\Psi_1(x) \vdash \Psi_0(x)$ and for every $\bar{m} \in A$ the following holds for $t = 0, 1$.

(4) *If* $\chi_t(x; \bar{m})$ & $\Psi_1(x)$ is consistent and $p(x) \vdash \neg\chi_t(x; \bar{m})$ *then* $R - M(\chi_t(x; \bar{m}); \Phi) \geqq (n, k-1)$.

Now we can prove the lemma in this case. Let $\Psi(x)$ be any formula from $p(x)$. We define by induction on $i < \omega$ formulas $\Psi^i(x) \in p(x)$ and $\bar{m}_i \in A$ such that

(a) $p(x) \vdash \neg\chi'(x; \bar{m}_i)$, $\chi'(x; \bar{m}_i)$ & $\Psi^i(x)$ is consistent,

(b) $\Psi^0(x) = \Psi(x)$ & $\Psi_1(x)$,

(c) $\Psi^{i+1}(x) = \Psi^i(x)$ & $\neg\chi'(x; \bar{m}_i)$, and

(d) $R - M(\chi'(x; \bar{m}_i); \Phi) = (n, k-1)$.

The definition is straightforward by the definition of (n, k) and the choice of Ψ_0. Choose $\chi(x; \bar{z}) \in L$ and $\bar{m}_i \in A$ for $i < \omega$ so that $\chi(x; \bar{m}_i) = \Psi_1(x)$ & $\chi'(x; \bar{m}_i)$. We have to prove that $\chi(x; \bar{m}_i) \vdash \neg\chi(x; \bar{m}_j)$ for $i > j$. Suppose not. Then we have that $\chi_0(x; \bar{m}_i{}^\frown\bar{m}_j)$ & $\Psi_1(x)$; $\chi_1(x; \bar{m}_i{}^\frown\bar{m}_j)$ & $\Psi_1(x)$ are both consistent, and so by (4) we have

$$R - M(\chi_t(x; \bar{m}_i{}^\frown\bar{m}_j); \Phi) \geqq (n, k-1) \qquad \text{for } t = 0, 1.$$

But $\chi_t \in L(\Phi)$, so we get $R - M(\chi'(x; \bar{m}_i); \Phi) \geqq (n, k)$, contradicting (d).

Case 2. $k = 1$. Then by (α) (and the definition of $R - M$), we have $n \geqq 1$. Once more there is $\Psi_0(x) \in p(x)$ such that whenever $\bar{m} \in A$ satisfies (2), (3) with Ψ replaced by Ψ_0 then $R - M(\chi'(x; \bar{m}); \Phi) \geqq (n-1, 1)$. Consider

$$\chi_3(x; \bar{z}') = \chi'(x; \bar{z}_0) \,\&\, \chi'(x; \bar{z}_1) \,\&\, \neg\chi'(x; \bar{z}_2),$$

where we assume that $\bar{z}_0$, $\bar{z}_1$, $\bar{z}_2$ are disjoint and $\bar{z}'$ is their concatenation. By the minimality of (n, k) there is $\Psi_1(x) \in p(x)$ such that $\Psi_1(x) \vdash \Psi_0(x)$ and for every $\bar{m} \in A$ the following holds:

(5) *If* $\chi_3(x; \bar{m}) \,\&\, \Psi_1(x)$ *is consistent and* $p(x) \vdash \neg\chi_3(x; \bar{m})$ *then* $R - M(\chi_3(x; \bar{m}); \Phi) \geqq (m-1, 1)$.

Now let $\Psi(x)$ be any formula from $p(x)$. Let us define $\Psi^i(x)$ and $\bar{m}_i \in A$ for $i < \omega$ so that (a), (b), (c) and

(d′) $(n, 1) > R - M(\chi'(x; \bar{m}_i); \Phi) \geqq (n-1, 1)$

hold. By properties of $R - M$ rank we can choose an increasing sequence $\langle i_s : s < \omega \rangle$ such that

(6) $R - M(\chi'(x; \bar{m}_{i_s}) \,\&\, \chi'(x; \bar{m}_{i_v}) \,\&\, \neg\chi'(x; \bar{m}_{i_r}); \Phi) < (n-1, 1)$ for every $s < \omega$, and $v, r > s$.

Choose $\chi(x; \bar{z}) \in L$ and $\bar{m}_s \in A$ so that for $s < \omega$,

(7) $\chi(x; \bar{m}_s) = \Psi_1(x) \,\&\, \chi'(x; \bar{m}_{i_{s+1}}) \,\&\, \neg\chi'(x; \bar{m}_{i_s})$.

Because of (a), (b), (c), (d′), for every $s < \omega$, $\Psi(x) \,\&\, \chi(x; \bar{m}_s)$ is consistent. Suppose that $s < v$. We shall prove that $\chi(x; \bar{m}_s) \,\&\, \chi(x; \bar{m}_v)$ is inconsistent. Because of (7) we can assume that $v > s+1$. Let

$$\theta_1(x) = \Psi_1(x) \,\&\, \chi'(x; \bar{m}_{i_{s+1}}), \quad \theta_2(x) = \Psi_1(x) \,\&\, \chi'(x; \bar{m}_{i_v}),$$

$$\theta_3(x) = \Psi_1(x) \,\&\, \chi'(x; \bar{m}_{i_{v+1}}).$$

By (5) and (6) we have $\theta_1(x) \,\&\, \theta_3(x) \,\&\, \neg\theta_2(x)$ is inconsistent. This means that $\chi(x; \bar{m}_s) \,\&\, \chi(x; \bar{m}_v)$ is inconsistent. Thus we have proved the lemma.

Lemma 2.5 justifies the following definition. For $\varphi(x; \bar{y}), \chi(x; \bar{z}) \in L$ we define $N(\varphi, \chi) = \{ p \in N(\varphi)$: for every formula $\Psi(x) \in p(x)$ there are $\bar{m}_i \in A$ for $i < \omega$ such that $\chi(x; \bar{m}_i) \,\&\, \Psi(x)$ is consistent and for $i \neq j$, $\chi(x; \bar{m}_i) \vdash \neg(x; \bar{m}_j)\}$. So by Lemma 2.5 we have

(γ) $N(\varphi) = \bigcup \{N(\varphi, \chi) : \chi \in L\}$.

Now the proof of Theorem 2.3 splits into two cases, depending on which of (A), (B), (C) from the statement of Theorem 2.2 holds.

Case I. $|T| < \operatorname{cov} \mathbf{K}$.

Lemma 2.6. *For every consistent formula* $\theta'(x) \in L(A)$ *with* $\theta'(x) \vdash \theta(x)$, *there are consistent formulas* $\varphi_n(x) \in L(A)$ *for* $n < \omega$ *such that* $\varphi_n(x) \vdash \theta'(x)$, *for* $n \neq m$; $\varphi_n(x) \vdash \neg\varphi_m(x)$; *and for every* $\varphi, \chi \in L$ *there is* $n < \omega$ *such that* $N(\varphi, \chi) \cap [\varphi_n] = \varnothing$.

PROOF. Suppose not. We shall construct formulas $\psi_n(x; \bar{y}_n) \in L(A)$ for $n > 0$ and parameters $\bar{m}_\eta \in A$ for $\eta \in {}^{\omega>}\omega$ such that the following hold.

(1) $\psi_{|\eta|}(x; \bar{m}_\eta)$ is consistent and $\psi_{|\eta|}(x; \bar{m}_\eta) \vdash \theta'(x)$,

(2) for $\eta \lessdot \nu \in {}^{\omega>}\omega$, $\psi_{|\nu|}(z; \bar{m}_\nu) \vdash \psi_{|\eta|}(x; \bar{m}_\eta)$, and

(3) if $\eta, \nu \in {}^{\omega>}\omega$ are incomparable then $\psi_{|\nu|}(x; \bar{m}_\nu) \vdash \psi_{|\eta|}(x; \bar{m}_\eta)$.

How to construct such a tree? First, there are φ, χ such that $N(\varphi, \chi) \cap [\theta'] \neq \varnothing$. Otherwise by (α) we could choose $\varphi_n(x)$, $n < \omega$, easily (and also (β) and the theorem would be proved). So we can choose ψ_1. Further on the construction of $\psi_n(x; \bar{y}_n)$ and $\bar{m}_\eta$, $\eta \in {}^{\omega>}\omega$, relies on the definition of $N(\varphi, \chi)$. Clearly the existence of such a tree contradicts the superstability of T, so we get a contradiction.

Using Lemma 2.6 we can conclude the proof of Theorem 2.3 in this case. We can construct a tree of formulas $\{\varphi_\eta(x) : \eta \in {}^{\omega>}\omega\} \subseteq L(A)$ such that

(a) $\varphi_\varnothing(x) = \theta(x)$,

(b) for $\eta \lessdot \nu$, $\varphi_\nu(x) \vdash \varphi_\eta(x)$, $\varphi_\eta(x)$ is consistent,

(c) if $\eta, \nu \in {}^{\omega>}\omega$ are incomparable then $\varphi_\nu(x) \vdash \neg\varphi_\eta(x)$, and

(d) for every $\eta \in {}^{\omega>}\omega$, $\varphi, \chi \in L$, there is $n < \omega$ such that $N(\varphi, \chi) \cap [\varphi_{\eta^\frown\langle n\rangle}] = \varnothing$.

For $\varphi, \chi \in L$ let

$$N'(\varphi, \chi) = \{f \in {}^\omega\omega\text{: there is } p \in N(\varphi, \chi) \text{ such that for every } n < \omega,\ p(x) \vdash \varphi_{f \restriction n}(x)\}.$$

By (d) we see that $N'(\varphi, \chi)$ is nowhere dense. So finally we can use the assumption that $|T| < \operatorname{cov} \mathbf{K}$. Indeed, as $|L| < \operatorname{cov} \mathbf{K}$, we have ${}^\omega\omega \neq \bigcup\{N'(\varphi, \chi) : \varphi, \chi \in L\}$ and that means that (β) and the whole theorem is proved in this case.

Case II. $|T| < \mathfrak{b}$ or $|T| < \min\{\operatorname{cov} \mathbf{L}, \mathfrak{b}\}$. Here the proof will be somewhat more complicated. We have to reformulate Lemma 2.6.

LEMMA 2.7. *Assume that* $|T| < \mathfrak{b}$ *or* $|T| < \operatorname{cov} \mathbf{L}$. *Then for every consistent formula* $\theta'(x) \in L(A)$ *with* $\theta'(x) \vdash \theta(x)$, *there are consistent formulas* $\varphi_n(x) \in L(A)$ *for* $n < \omega$ *such that* $\varphi_n(x) \vdash \theta'(x)$, *for* $n \neq m$; $\varphi_n(x) \vdash \neg\varphi_m(x)$; *and for every* $\varphi, \chi \in L$, *for all but finitely many* $n < \omega$ *we have* $N(\varphi, \chi) \cap [\varphi_n] = \varnothing$.

PROOF. (1) First assume that $|T| < \mathfrak{b}$. Let $\forall^\infty$, $\exists^\infty$ mean "for all but finitely many" and "there are infinitely many", respectively. We shall construct by induction on $k<\omega$ formulas $\psi_k(x;\bar{y}_k)\in L$ and parameters $\{\bar{m}^k_\eta : \eta\in {}^{k\geq}\omega\}\subseteq A$ so that

(1) $\psi_0(x;\bar{m}^k_\emptyset)=\theta'(x)$, $\psi_{|\eta|}(x;\bar{m}^k_\eta)$ is consistent,

(2) if $\eta\lessdot\nu\in {}^{k\geq}\omega$ then $\psi_{|\nu|}(x;\bar{m}^k_\nu)\vdash\psi_{|\eta|}(x;\bar{m}^k_\eta)$, and

(3) if $\eta,\nu\in {}^{k\geq}\omega$ are incomparable then $\psi_{|\nu|}(x;\bar{m}^k_\nu)\vdash\neg\psi_{|\eta|}(x;\bar{m}^k_\eta)$.

For $k=1$ we can find ψ_1 and $\{\bar{m}^1_\eta : \eta\in {}^{1\geq}\omega\}$ by Lemma 2.5, similarly as in the proof of Lemma 2.6 (as otherwise Lemma 2.7, as well as the theorem, would be proved). Suppose that we have constructed $\psi_0,\dots,\psi_k$ and $\{\bar{m}^k_\eta : \eta\in {}^{k\geq}\omega\}$. Suppose also that for some $\varphi,\chi\in L$,

(4) $\exists^\infty n_0\,\exists^\infty n_1\cdots\exists^\infty n_{k-1}$ (if $\eta=\langle n_0,\dots,n_{k-1}\rangle$ then $[\psi_k(x;\bar{m}^k_\eta)]\cap N(\varphi,\chi)\neq\emptyset$).

Then by the definition of $N(\varphi,\chi)$ we can find ψ_{k+1} and $\{\bar{m}^{k+1}_\eta : \eta\in {}^{k+1\geq}\omega\}$ such that (1), (2), (3) hold for $k+1$. However, if we really managed to carry out this construction for every $k<\omega$, then it would contradict the superstability of T (by compactness). Thus for some k such that $\psi_0,\dots,\psi_k$ and $\{\bar{m}^k_\eta : \eta\in {}^{k\geq}\omega\}$ still satisfy (1), (2) and (3), we have for every $\varphi,\chi\in L$:

(5) $\forall^\infty n_0\,\forall^\infty n_1\cdots\forall^\infty n_{k-1}$ (if $\eta=\langle n_0,\dots,n_{k-1}\rangle$ then $[\psi_k(x;\bar{m}^k_\eta)]\cap N(\varphi,\chi)=\emptyset$).

Fix $\varphi,\chi\in L$. By (5) we can define $n(\varphi,\chi)<\omega$ and functions $g_i(\varphi,\chi): {}^i\omega\to\omega$ for $i=1,\dots,k-1$ such that if $\eta\in{}^k\omega$ satisfies $\eta(0)>n(\varphi,\chi)$ and $\eta(i)>g_i(\varphi,\chi)(\eta\restriction i)$ for $i>0$ then

$$[\psi_k(x;\bar{m}^k_\eta)]\cap N(\varphi,\chi)=\emptyset.$$

As $|T|<\mathfrak{b}$, we can find functions $g_1,\dots,g_{k-1}\in{}^\omega\omega$ such that for every $\varphi,\chi\in L$,

$$g_i(\varphi,\chi)(\langle\cdot,g_1(\cdot),\dots,g_{i-1}(\cdot)\rangle)\prec^{*} g_i \qquad \text{for } i>0.$$

For $n<\omega$ we define $\eta_n=\langle n,g_1(n),\dots,g_{k-1}(n)\rangle$, and let $\varphi_n(x)=\psi_k(x;\bar{m}^k_{\eta_n})$. We shall check that φ_n, $n<\omega$, satisfy our requirements. So let $\varphi,\chi\in L$. Take $n<\omega$ so large that for $i>0$,

$$g_i(\varphi,\chi)(\langle n,g_1(n),\dots,g_{i-1}(n)\rangle)<g_i(n) \quad\text{and}\quad n>n(\varphi,\chi).$$

By the choice of functions $g_i(\varphi,\chi)$ and $n(\varphi,\chi)$ we see that $[\varphi_n]=[\psi_k(x;\bar{m}^k_{\eta_n})]$ is disjoint to $N(\varphi,\chi)$, so the lemma is proved in this case.

(2) Now assume that $|T|<\operatorname{cov}\mathrm{L}$. Let $\exists^{\geq k}$ mean "there are $\geq k$-many". for $0<k$, $n<\omega$ let T^k_n be the tree ${}^{k\geq}[2^n,2^{n+1})$. We shall construct for

$0<k<\omega$ formulas $\psi_k(x;\bar{y}_k)\in L$ and parameters $\{\bar{m}^k_\eta : \eta\in T^k_n,\ \text{some } n\}$ such that:

(1) $\psi_{|\eta|}(x;\bar{m}^k_\eta)\vdash\theta'(x)$ and $\psi_{|\eta|}(x;\bar{m}^k_\eta)$ is consistent,

(2) if $\eta\lessdot\nu$ then $\psi_{|\nu|}(x;\bar{m}^k_\nu)\vdash\psi_{|\eta|}(x;\bar{m}^k_\eta)$, and

(3) if η,ν are incomparable then $\psi_{|\nu|}(x;\bar{m}^k_\nu)\vdash\neg\psi_{|\eta|}(x;\bar{m}^k_\eta)$.

As before we can easily find ψ_1 and $\{\bar{m}^1_\eta : \eta\in T^1_n,\ \text{some } n\}$. Suppose that we have constructed $\psi_1,\ldots,\psi_k$ and $\{\bar{m}^k_\eta : \eta\in T^k_n,\ \text{some } n\}$, so that (1), (2), (3) hold. Suppose that for some $\varphi,\chi\in L$, for every $K<\omega$ there is $n<\omega$ such that

(4) $(\exists^{\geq K}n_0\ \exists^{\geq K}n_1\cdots\exists^{\geq K}n_{k-1})$ (if $\eta=\langle n_0,\ldots,n_{k-1}\rangle$ then $\eta\in T^k_n$ and $[\psi_k(x;\bar{m}^k_\eta)]\cap N(\varphi,\chi)\neq\emptyset$).

If (4) holds, then by the definition of $N(\varphi,\chi)$ we can find ψ_{k+1} and $\{\bar{m}^{k+1}_\eta : \eta\in T^{k+1}_n,\ \text{some } n\}$, so that (1), (2), (3) hold for $\psi_1,\ldots,\psi_{k+1}$. However, if we really manage to carry out this construction for $k<\omega$, then by the compactness theorem we get a contradiction with the superstability of T. So there is $k>0$ such that for every $\varphi,\chi\in L$ there is $K(\varphi,\chi)<\omega$ such that for every $n<\omega$, $\neg$(4) holds with K replaced by $K(\varphi,\chi)$.

Let $S=\Pi_{0<n<\omega}\,{}^k[2^n,2^{n+1})$. On ${}^k[2^n,2^{n+1})$ we define a measure μ_n by $\mu_n(\{\eta\})=1/2^{nk}$ for $\eta\in{}^k[2^n,2^{n+1})$, and let μ be the product measure of μ_n, $n<\omega$ on S. For $\varphi,\chi\in L$ let us define

$$N_0(\varphi,\chi)=\{f\in S:\ \exists^\infty n[\psi_k(x;\bar{m}^k_{f(n)})]\cap N(\varphi,\chi)\neq\emptyset\}.$$

The following claim is easy, so we omit its proof.

Claim. $\mu(N_0(\varphi,\chi))=0$.

Now we can use the assumption that $|T|<\operatorname{cov}\mathbf{L}$. There is $f\in S$ such that for every $\varphi,\chi\in L$, $f\notin N_0(\varphi,\chi)$. Let $\varphi_n(x)=\psi_k(x;\bar{m}^k_{f(n)})$. We see that for every $\varphi,\chi\in L$, there are only finitely many $n<\omega$ such that $[\varphi_n]\cap N(\varphi,\chi)\neq\emptyset$, so the lemma is proved.

Now we can finish the proof of Theorem 2.3. By Lemma 2.7 we can construct a tree of formulas $\{\varphi_\eta(x):\eta\in{}^{\omega>}\omega\}\subseteq L(A)$ such that

(a) $\varphi_\emptyset(x)=\theta(x)$,

(b) for $\eta\lessdot\nu$, $\varphi_\nu(x)\vdash\varphi_\eta(x)$, $\varphi_\eta(x)$ is consistent,

(c) if η,ν are incomparable then $\varphi_\nu(x)\vdash\neg\varphi_\eta(x)$, and

(d) for every $\eta\in{}^{\omega\geq}\omega$, $\varphi,\chi\in L$, for all but finitely many $n<\omega$, we have $N(\varphi,\chi)\cap[\varphi_{\eta^\frown\langle n\rangle}]=\emptyset$.

Let $N'(\varphi,\chi)$ be defined as in Case I. By (d) we see that for every $\varphi,\chi\in L$, $\operatorname{cl}(N'(\varphi,\chi))$ is compact, i.e. there is $g(\varphi,\chi)\in{}^\omega\omega$ such that for every $f\in N'(\varphi,\chi)$

we have $f\text{-}\exists g(\varphi, \chi)$. The assumptions of Case II imply that $|T| < \mathfrak{d}$, so we can choose $g \in {}^{\omega}\omega$ such that for every $\varphi, \chi \in L$ we have $\neg g\text{-}\exists g(\varphi, \chi)$. This means that $g \notin N'(\varphi, \chi)$ for any φ, χ, and so (β) holds and Theorem 2.3 is proved.

Let us draw corollaries from Theorem 2.2 (and Fact 2.1).

COROLLARY 2.8. *Every superstable T of power*

$$< \mathfrak{b} + \operatorname{cov} \mathbf{K} + \min\{\operatorname{cov} \mathbf{L}, \mathfrak{d}\}$$

has the extension property and model extension property.

Corollary 2.8 shows that the powers of T_0 and T_1 from §1 cannot be smaller that $2^{\aleph_0}$ in ZFC only. T_0 [T_1] is a "minimally complicated" theory without [model] extension property, yet from another point of view. We have $D(T_0) = 2$ and $D(T_1) = 3$. We can prove

FACT 2.9. (1) If T is superstable and $D(T) = 1$, then T has the extension property.

(2) If T is superstable and $D(T) \leqq 2$, then T has the model extension property.

PROOF. We shall prove only (2), as (1) is easier. So let $M \not\prec N$ be models of T with $Q(M) = Q(N)$. Clearly it suffices to prove the following.

(†) If $A \supseteq N$, $(Q(x), A)$ has $T\text{-}V$ property and $\theta(x) \in L(A)$ is consistent, then for some $a \in \theta(\mathfrak{C})$, $(Q(x), A \cup \{a\})$ has $T\text{-}V$ property.

First notice that if $\theta(\mathfrak{C}) \cap N \neq \varnothing$ then we are done. Otherwise $\theta(x)$ forks over N, so $D(\theta(x)) \leqq 1$. However, if there is no $a \in \theta(\mathfrak{C})$ such that $(Q(x), A \cup \{a\})$ has $T\text{-}V$ property, then Lemma 2.5 gives us an infinite uniform family of non-algebraic formulas below $\theta(x)$, so we have a contradiction.

Let us summarize the information on theories with extension property which we have obtained.

(a) *Stable case*. If $|T| = \aleph_0$ then T has extension property. There is T of power $\aleph_1$ without model extension property.

(b) *Superstable case*. If $|T| < \mathfrak{b} + \operatorname{cov} \mathbf{K} + \min\{\operatorname{cov} \mathbf{L}, \mathfrak{d}\}$ then T has extension property. There is T of power $\kappa_1 \leqq 2^{\aleph_0}$ without model extension property.

We have $\mathfrak{b} + \operatorname{cov} \mathbf{K} + \min\{\operatorname{cov} \mathbf{L}, \mathfrak{d}\} \leqq \kappa_1$, so one can ask what happens when $\mathfrak{b} + \operatorname{cov} \mathbf{K} + \min\{\operatorname{cov} \mathbf{L}, \mathfrak{d}\} \leqq |T| < \kappa_1$. The author suspects that in Theorem 2.2, $\mathfrak{b} + \operatorname{cov} \mathbf{K} + \min\{\operatorname{cov} \mathbf{L}, \mathfrak{d}\}$ can be replaced by $\mathfrak{d}$. We have the following partial result in this direction.

FACT 2.10. (1) If $|T| < \mathfrak{d}$, T is superstable and $D(T) \leqq 2$, then T satisfies (LA).

(2) If $|T| < \mathfrak{d}$, T is superstable and $D(T) \leqq 3$, then T has model extension property.

PROOF. (1) If it is not true, then for some $A \subseteq \mathfrak{C}$ and for some consistent $\theta(x) \in L(A)$, there is no locally isolated $p \in S(A) \cap [\theta]$. We keep the notation from the proof of Theorem 2.3. By Lemma 2.5, and by $D(\theta) \leqq 2$, we get that Lemma 2.7 holds in our case. The rest is easy.

(2) follows from (1) and the proof of Fact 2.9(2).

Although the problem of determining which cardinal can replace the present estimation in Theorem 2.2 is open, we have

COROLLARY 2.11. (1) Con(ZFC + "*every superstable* T *of power* $< 2^{\aleph_0}$ *satisfies* (LA), (EP) *and* (ME)" + $2^{\aleph_0}$ *large*).

(2) Con(ZFC + "*there is a superstable* T *of power* $\aleph_1$ *without model extension property*" + $2^{\aleph_0}$ *large*).

PROOF. (1) Cohen's forcing yields cov **K** and $2^{\aleph_0}$ large. (2) T_1' from Example 1 is a superstable theory of power κ_1 without model extension property. See for example [M1] or [N] on how to make κ_1 equal $\aleph_1$ while preserving $2^{\aleph_0}$ large.

In cases (A), (B), (C) the proof of Theorem 2.3 is increasingly complicated and relies more and more on the compactness theorem (Lemma 2.7). However, the results which we finally obtain are not stronger at all. If you take any two cardinals from {cov **K**, cov **L**, $\mathfrak{b}$} then you can find a model $\mathfrak{M}$ in which one of these cardinals equals $\aleph_1$ and the other $\aleph_2$. Let us note for example some of these well-known results (a wider exposition can be found in [K], [M1], [M2] or [N]):

(1) Con(ZFC + cov **K** > $\mathfrak{b}$) (forcing with ω_2 Cohen reals over a model of CH).

(2) Con(ZFC + cov **K** < $\mathfrak{b}$) (forcing with ω_1 random reals over a model of $2^{\aleph_0} = \aleph_2$ and MA).

(3) Con(ZFC + ($\mathfrak{b}$ + cov **K** < κ_1)) (see [M1] or [N]).

The following problem seems interesting. If we have a superstable T in some model $\mathfrak{M}$ of ZFC and in $\mathfrak{M}$, T does not have (model) extension property, then by adding to $\mathfrak{M}$ $|T|^+$-many Cohen reals we obtain a model $\mathfrak{N}$ in which T has extension property. Is the reverse process possible? I.e. isn't it so that if T has extension property in a model $\mathfrak{M}$ of ZFC then, for every $\mathfrak{N} \supseteq \mathfrak{M}$, T has extension property in $\mathfrak{N}$?

In Example 1, in modifications T_0' and T_1' of T_0 and T_1 we requested N_α's to be disjoint. The reader might wonder if this requirement could be omitted. One can see directly that if N_α's are not disjoint, then we lose superstability. Another way to see this is as follows. Suppose that we can prove in ZFC that for some collection of N_α's of power cov $\mathbf{K}$ covering the real line, the resulting T_0' and T_1' are superstable. Then this would hold in a model of ZFC + cov $\mathbf{K} < \mathfrak{b}$ (notice also that "being superstable' is absolute). In such a model, however, T_1' must have model extension property by Corollary 2.8 (because $|T_1'| = \text{cov}\,\mathbf{K}$ here), a contradiction.

The functions f_η, $\eta \in {}^\omega 2$ are the only reason why T_1 is uncountable. (Losing stability) we can deal with this problem as follows. Instead of having a distinct name for each function f_η, we can define them uniformly as $f_z(x)$, where the parameter z runs over a definable subset of Q. The resulting *countable* theory T satisfies the following. There are models $M \not\prec N$ with $Q(M) = Q(N)$ such that N is a conservative extension of M, and there is no $N' \not\succ N$ with $Q(N') = Q(N)$. This answers negatively a question of Baldwin from [B].

Recently the author has strengthened Theorem 2.2 by weakening the assumption that T is superstable to "T is stable and $\kappa(T) \leqq \aleph_1$".

References

[B] J. T. Baldwin, *Conservative extensions and the two-cardinal theorem for stable theories*, Fund. Math. **88** (1975), 7–9.

[H] V. Harnik, *A two-cardinal theorem for sets of formulas in a stable theory*, Isr. J. Math. **21** (1975), 7–23.

[K] K. Kunen, *Random and Cohen reals*, in *Handbook of Set-theoretical Topology* (K. Kunen and J. E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 887–911.

[L] A. H. Lachlan, *A property of stable theories*, Fund. Math. **77** (1972), 9–20.

[Ls] D. Lascar, *Types définissible et produit de types*, C.R. Acad. Sci. Paris **276** (1973), 1253–1256.

[M1] A. W. Miller, *Covering 2^ω with ω_1 disjoint closed sets*, The Kleene Symp. Proc. Symp. Univ. Wisconsin, 1978, Studies in Logic and Found. Math. **101**, North-Holland, Amsterdam, 1980, pp. 415–421.

[M2] A. W. Miller, *Some properties of measure and category*, Trans. Am. Math. Soc. **266** (1981), 93–114.

[N] L. Newelski, *On partitions of the real line into compact sets*, J. Symb. Logic **52** (1987), 353–359.

[S] S. Shelah, *Classification Theory*, North-Holland, Amsterdam, 1978.